THE PLANIMETER OF J. GIERER
Invented by Misunderstanding

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As a software engineer in research and development he worked on topics as diverse as room acoustics, numerical simulation of electric and magnetic fields, wave propagation, field-strength prognosis for broadcasters and degaussing of ships.

As a collector of calculation devices he is especially interested in analog computing mechanisms such as planimeters, nomograms etc.

Preface

In the middle of the 19th century, land surveying and cadastral mapping was introduced in most European countries. Fiscal laws required size and value of land areas to be determined authoritatively as a basis for tax equity. All real estate had to be recorded and its extent, value and ownership documented. The need for precise calculation of area especially for taxation purposes led to the appearance of various measuring devices, and a variety of planimeters was invented.

In 1853, the German professor of geodesy, Carl Maximilian von Bauernfeind, published the first general overview of the state of the art of planimetry in which he described the many different devices and their functional principles. At that time, the devices in use ranged from gadgets as simple as harp planimeters, which give only an approximate value of the area, to "real" planimeters that determine the area precisely by tracing a pointer around its perimeter.

Johann Andreas Gierer, a drawing teacher from the Franconian town Fürth, was fascinated by Bauernfeind's article, especially by a device called "Ringmesser" (circle gauge). Since this planimeter was only rudimentarily described by Bauernfeind, Gierer decided to re-invent it. The fruit of his inventive talent was not a simple approximating instrument like the Ringmesser, but a real integrating planimeter incorporating a unique functional principle.

Early Planimeters

The bestiary of planimeters can roughly be divided into two classes: non-integrating planimeters and integrating planimeters.

The first class, non-integrating planimeters, determine the area in question either by decomposing it into parts whose area can easily be calculated (e. g. triangles), or by approximating the area with a number of small calculable "finite elements" and adding up their sizes. The simplest (but inexact) method to approximate the area of an irregularly shaped figure is to cover it with a grid of equal-sized rectangles and to count them. The same result can be achieved more easily by dividing the figure into stripes of equal width and adding up their lengths. This is the principle of the various forms of harp planimeters.

On the other hand an integrating planimeter determines the area exactly via integral calculus. The first integrating planimeter was invented in 1814 by the Bavarian land surveyor Johann Martin Hermann. Hermann created a multiplying gear by combining a recording wheel with a spinning cone. A carriage, in which the cone is mounted, moves along the x-axis and causes the cone to rotate. The recording wheel can be moved parallel to the y-axis thus rotating the faster the nearer it is to the cones base. The recorded value is proportional to the distance covered along the x-axis, as well as to the position of the wheel on the cone, which is the y-value of the function traced by a cursor steering the wheel.

About 10 years later, the Italian mathematician and Professor Tito Gonella, in ignorance of Hermann's ideas, re-invented the cone-wheel mechanism. What is more, he realised that the working of the mechanism did not depend on the opening angle of the cone. It could be done as well with an angle as wide as 180°, leading to the replacement of the cone by a disc and the invention of the disc-wheel integrating mechanism.

The cone-wheel-mechanism experienced its third re-invention in 1827 by the Swiss inventor Johannes Oppikofer, and its fourth re-invention by John Sang in 1851.

Bauernfeind's article

The German geodesist Carl Maximilian von Bauernfeind (28.11.1818 - 3.8.1894) can be regarded as one of the founders of geodesy as a modern science. In his article "*Die Planimeter von Ernst, Wetli und Hansen, welche den Flächeninhalt ebener Figuren durch das Umfahren des Umfangs angeben*", Bauernfeind describes the state of the art of planimetry as far as it is known to him. In particular, the planimeters of Hermann and Gonella were not yet known to Bauernfeind in 1853 when the article was published. The article specifies the various forms of non-integrating and integrating planimeters and presents the underlying mathematical theory.

Bauernfeind limits himself to short notes concerning non-integrating planimeters, such as the method of counting squares using graph paper, or the Oldendorp planimeter, which measures stripes. One of the described non-integrating planimeters is the so called "*Westfeld'sche Ringmesser*", Westfeld's circle gauge. On the 47 pages of Bauernfeind's article only a few lines mention this Ringmesser, and there is only one sentence about its functional principle. But since a description of the Ringmesser had already been published in 1826, Bauernfeind preferred to direct the reader's attention to the newer, integrating planimeters.

The Planimeters of Ernst, Wetli and Hansen

As the title implies, the major part of Bauernfeinds article treats the planimeters of Ernst, Wetli and Hansen. Because these planimeters make use of two independent movements orthogonal to each other, their type is also called orthogonal planimeter.

Heinrich Rudolf Ernst came in contact with Oppikofers planimeter in Switzerland, where he was charged with finishing its construction in 1828 after the death of the original instrument maker [Fisc2002]. Later, in 1835 in Paris, he constructed his own planimeter (Fig. 1), which was based on the same principle, the cone-wheel-mechanism.

In 1849, Caspar Wetli from Switzerland designed a coordinate planimeter which used a disc-wheel-gear to calculate the area. This is the first occurence of the disc-wheel-mechanism after Gonellas planimeter from 1823, which was obviously unknown to Wetli. Nevertheless, in his article Bauernfeind points out the fact that the basic principle of Wetlis planimeter is the same as that of Ernst's, and he regards Wetli's device not as an independent invention but an improvement of the Ernst type.

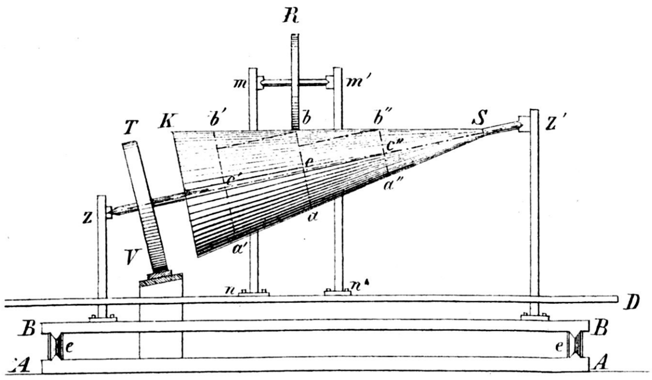


Fig. 1 - The Ernst planimeter [Baue1853]

The third orthogonal planimeter that Bauernfeind mentions in the title of his article and gives an in-depth treatment to, was constructed in 1851 by the German astronomer Peter Andreas Hansen. Hansen's planimeter (Fig. 2), is based on Wetli's disc-wheel-mechanism with several practical improvements.

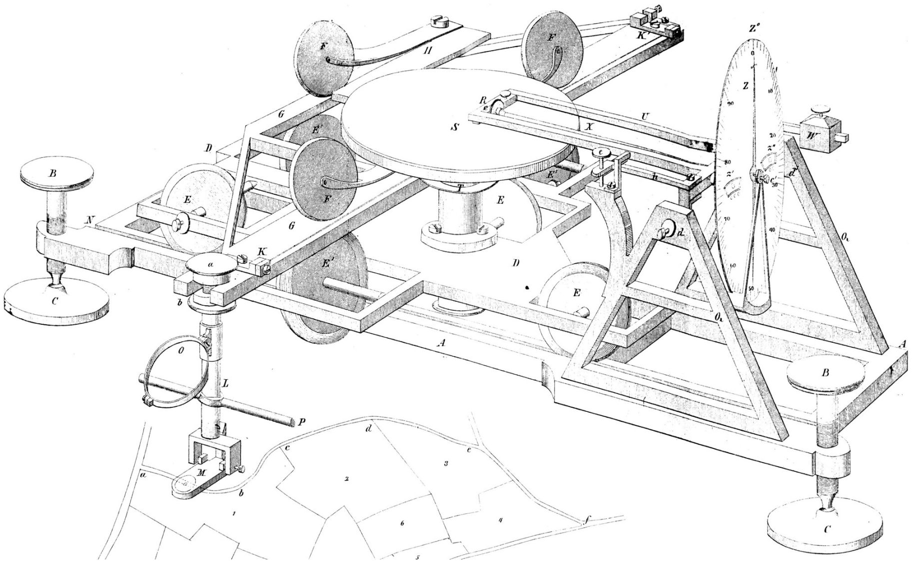


Fig. 2 - The Hansen planimeter [Baue1853]

How does the Orthogonal Coordinate Planimeter work?

The most basic recipe for measuring an area is: Take a small rectangle with a known area and count how many of those rectangles fit into the region of which you want to know the area. That is what you do when estimating the area with the use of graph paper. This type of non-integrating planimeter works according to the formula:

$$A = \sum_i \Delta x \Delta y$$

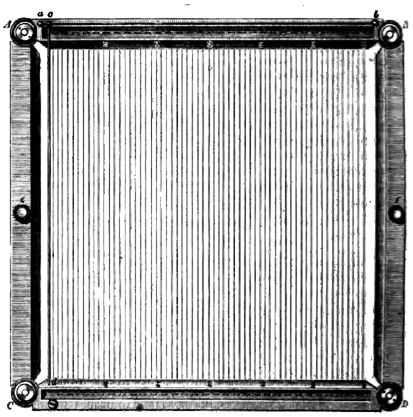


Fig. 3 - The Oldendorp planimeter measures an area by summing up the lengths of parallel stripes [Hunä1864]

Of course this is only an approximation, but you can get a better result by decreasing the size of the rectangles. The best result, i. e. an exact measurement of the area, will be achieved when the size of the rectangles becomes infinitely small:

$$A = \int \mathrm{d}x \, \mathrm{d}y$$

There is no obvious way to handle infinitely small areas in practice, and even counting small rectangles is too much work. To speed things up, you can use a trick. Arrange all the small rectangles in parallel rows. Instead of counting the small rectangles of a single row, you can measure the length of this stripe. One of this type of stripe planimeters, the Oldendorp planimeter, was also mentioned by Bauernfeind.

The Oldendorp planimeter is a tool for dividing an area into stripes. The lengths of these stripes can be added with a pair of compasses, according to the formula:

$$A = \sum_i y_i \Delta x$$

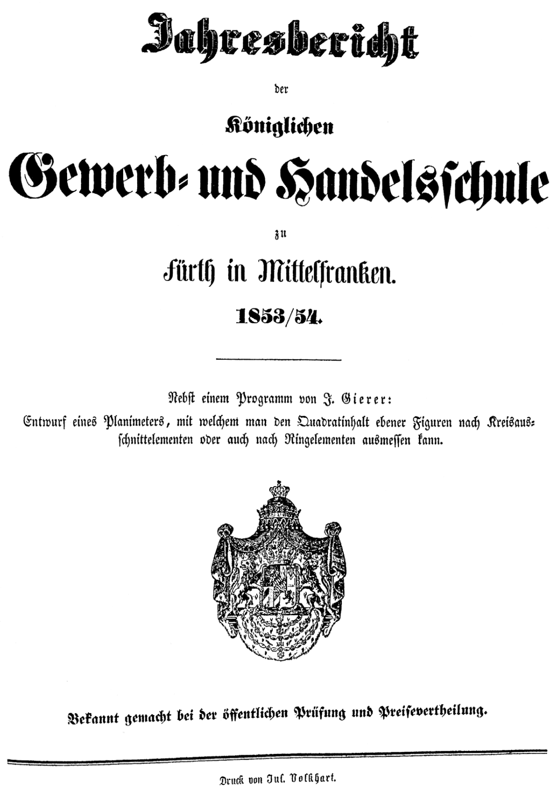
Again, this is an approximation. To obtain the exact area, the width of the stripes must be made infinitely small:

$$A = \int y \, \mathrm{d}x$$

A mechanism working according to this formula will have to measure the lengths of the stripes, multiply them by the infinitesimally small width and to sum them up. There is a quite common device that determines the sum of infinitesimal magnitudes. We all know the mileage-counter in a car, which counts the distance covered of a wheel on a surface. For measuring areas instead of mileages we need to connect the "mileage"-counter with the measurement of the stripe lengths in a manner that the counter counts faster, when the stripe is longer. That means we need a "car" with a "mileage"-counter and a continuous gear box. As the continuous gear box one can use a cone-wheel-mechanism or a disc-wheel mechanism. The latter is just a very obtuse cone, so in theory both are the same.

The Ernst planimeter with its cone-wheel-mechanism and the Hansen planimeter with its disc-wheel-mechanism are both based on the same theory.

Bauernfeind's planimeter article was not the first, but it was one of the more important and widespread articles read by many people. One of them was Johann Andreas Gierer.


Jahresbericht
der
Königlichen
Gewerb- und Handelsschule
zu
Fürth in Mittelfranken.
1853/54.

Nebst einem Programm von J. Gierer:
Entwurf eines Planimeters, mit welchem man den Quadratinhalt ebener Figuren nach Kreisausschnittelementen oder auch nach Ringelementen ausmessen kann.

Bekannt gemacht bei der öffentlichen Prüfung und Preisevertheilung.

Druck von Jul. Volkhart.

Fig. 4 - Gierer's publication on his planimeter [Gier1824]

Johann Andreas Gierer

Not much is known about the life of Johann Andreas Gierer (15.6.1798 - 2.5.1864). He was a teacher for draftsmanship at the "Gewerb- und Handelsschule", the vocational school, in Fürth, a town next to Nürnberg in the kingdom of Bavaria. This school was founded in 1833, and first used rooms in the inn "Zum Roten Roß", but moved later to another building. Gierer was one of their first teachers. The area around Fürth and Nürnberg was at that time an booming industrial region. In 1835, Germany's first railway was built to connect both towns. It was a time when drawing was very close to both drafting and painting. Consequently, Gierer tought both freehand drawing and technical drawing. [Vett1856] [Vett1864] [Fron1887] [Schw1968]

Gierer's Intention

Gierer was a member of the "Polytechnischer Verein für das Königreich Bayern", the Polytechnical Society for the Kingdom of Bavaria. He was a regular reader of the "Kunst- und Gewerbeblatt", the journal of that society. It was the issues of March and April 1853 of this journal where Bauernfeind first published his article [Bau1853]. Through this article Gierer was introduced to planimeters, mainly orthogonal integrating planimeters. He became curious about variants of these planimeters, especially about Westfeld's Ringmesser. Gierer had an idea:

"… [Westfelds Ringmesser] gab, da ich diese Beschreibung nicht besitze, Veranlassung zu untersuchen, ob es mir nicht möglich wäre, einen Planimeter zu entwerfen, der (...) nach Elementen von Ringstücken oder Kreisausschnitten mißt." ([Westfelds Ringmesser] , since I do not have its description, induced me

to examine, if it where not possible, to design a planimeter, which measures using elements of rings or sectors of a circle.) [Gier1854]

Gierer tried to reconstruct the Ringmesser of Westfeld on this sparse information about that device *"it is based on something circular"*, but retaining the assumption, that it had the same purpose as an orthogonal integrating planimeter, which is described in detail in that article.

Instead of using rectangular elements $\Delta x \Delta y$, Gierer intended to use ring elements $\Delta r \Delta b$, where Δr is the difference of the inner and outer radius of a ring, and Δb is the length of the arc of this ring element. In polar coordinates, we prefer to use angles instead of arc lengths for the small area element:

$$A = \sum_i \Delta r \, r_i \, \Delta\varphi$$

Or, for an exact rather than an approximate solution:

$$A = \int r \, \mathrm{d}r \, \mathrm{d}\varphi$$

As in the case to stripes instead of rectangles, we can save a lot of work by summing up sectors instead of ring elements. We have to keep in mind that a sector with a very small angle is more like a triangle than a rectangle. Therefore we need a factor of ½ in the following formula:

$$A = \sum_i \frac{1}{2} r_i^2 \, \Delta\varphi$$

Thereby, we gain as exact formula:

$$A = \frac{1}{2} \int r^2 \mathrm{d}\varphi$$

That is what Gierer intended to use as the underlying theory for his planimeter. Instead of the length of a stripe, he measures the radius. Since the radius is a length, too, this is not difficult to do. Again, Gierer uses a cone-wheel-"milage"-counter to add up these infinitesimally small magnitudes. But there is an r^2 in the formula. That means Gierer had to invent a mechanism to square a magnitude.

The Squaring Mechanism

As described above, Gierer needed a mechanism with a radius as input and another length as output, the square of the input. Being a draftsman, Gierer started by drawing the input magnitude on the right side and the corresponding output magnitude on the left side as shown in fig. 5. A mechanism consisting of only two linkages is very simple, and that is what Gierer used: two linkages. He used one end of the first linkage as input and one end of the other linkage as output. The other ends of the both linkages were connected to each other with a hinge. If this mechanism is to give a defined output for a defined input, the connection point cannot be just anywhere but must move along a defined curve. With the known input and output, Gierer was able to fix the position of the intermediate connection by drawing the linkages for some input and output values. (In fact, Gierer drew only the endpoints of the linkages but in fig. 5 the linkages are shown.)

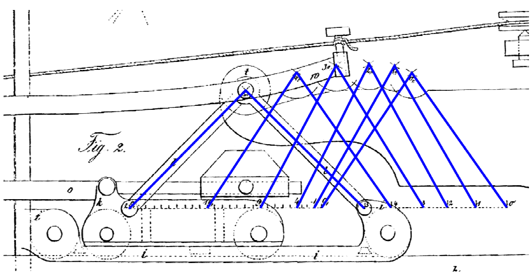


Fig. 5 - The linkage mechanism for squaring a length and its construction [Gier1854]

The required curve for the intermediade linkage connection point was interpolated by drawing a continuous line through the constructed points. To force the hinge to move along this curve, Gierer used an additional roll whose axis coincided with the hinge axis. That means that the center of the roll moves along the curve. That is achieved by rolling the roll along a path different from the curve but dependent on it.

The Blueprint

Gierer published an article on his findings in 1854. It included a drawing consisting two parts: a side view (Fig. 6) and a top view (Fig. 7) [Gier1854].

The planimeter consists of a base ring, on which three rollers revolve around the rings centre. These rollers support the framework and allow the whole device to turn. The cone of the cone-wheel-mechanism is mounted coaxially to one of these rollers, so the cone will rotate when the framework turns.

On the framework two carts can glide radially along two rails. These two carts are connected by the squaring mechanism. For the input magnitude a rod is fixed to the first cart and bears a cursor and a magnifying glass.

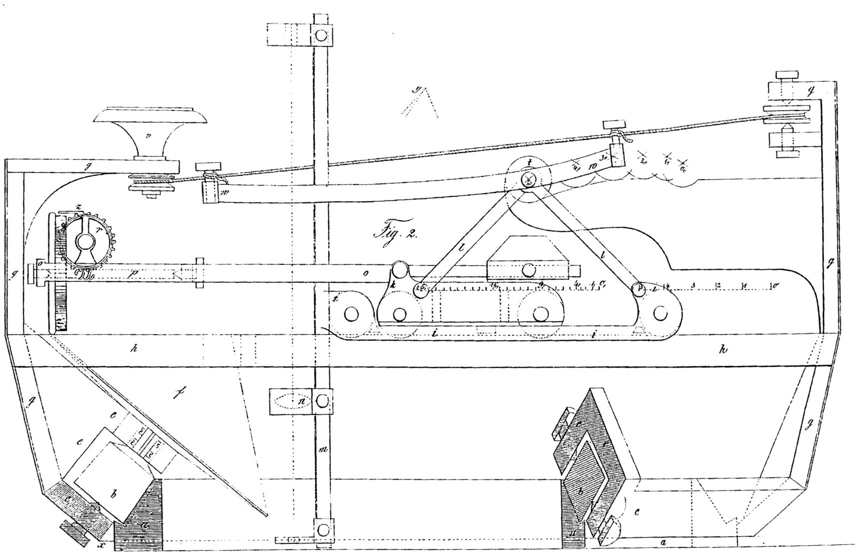


Fig. 6 - Side view of the Gierer planimeter [Gier1854]

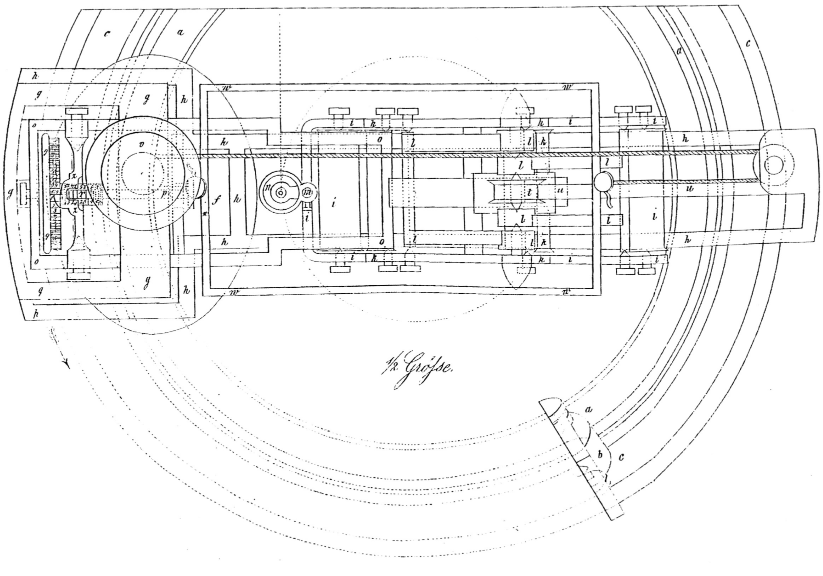


Fig. 7 - Top view of the Gierer planimeter [Gier1854]

When the device is used, the border of an area is traced with the cursor. Following the cursor, the framework turns around and the cart glides back and forth according to the radial distance. The two linkages of the squaring mechanism cause the second cart to move in a radial direction proportionally to the square of the radial distance. This movement is transferred to the wheel of the cone-wheel-mechanism (Fig. 9). This wheel is divided into one hundred sections. Each turn of the wheel is counted with a worm gear, and a second wheel that is divided into twenty five sections.

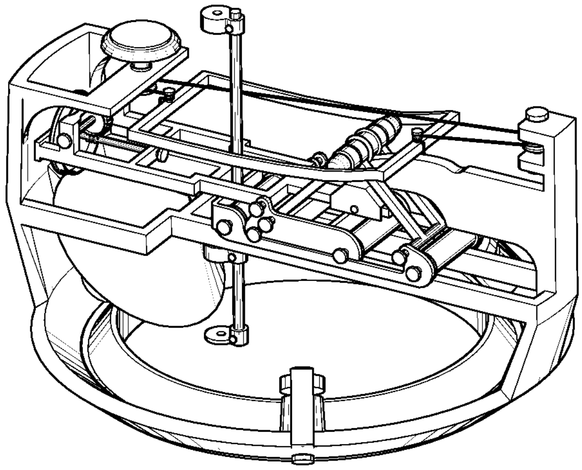


Fig. 8 - Perspective view of the Gierer planimeter

Gierer'**s** design does not propose to move the cursor directly. The cursor is moved by the framework in a circumferential direction. In radial direction the cursor and the carts are shifted by moving the hinge of the squaring mechanism. For this purpose there is a second smaller frame connected to the hinge, that can be handled with a wire by turning a knob fixed to the framework (Fig. 10). The path for the hinge of the squaring mechanism is also a part of the framework.

Gierer's planimeter

When measuring the area of a region, the planimeter (Fig. 8) is placed upon the map with the region under the planimeter. All the movements of the planimeter are controlled with the knob on the framework. By turning and moving the knob, the cursor is placed at a starting point at the border of the region. The operator has to note the reading of the counter mechanism. Using only knob operations, the boundary of the area is traced with the cursor until the starting point is reached again. Here the reading of the counter is noted again. The difference of the two readings is the area of the region.

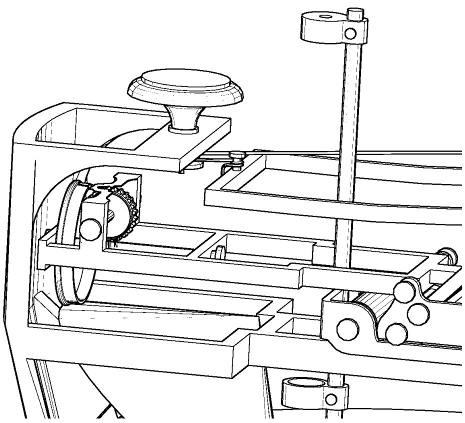


Fig. 9 - The cone-disc-mechanism of the Gierer planimeter

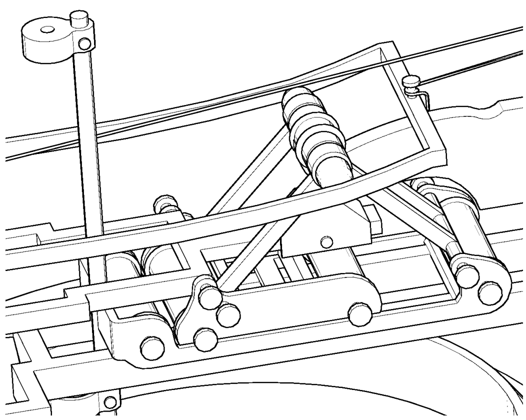


Fig. 10 - The squaring mechanism of the Gierer planimeter

Misunderstanding as the mother of invention

We have seen what Gierer did to reconstruct Westfeld's Ringmesser without having access to Westfeld's description. Now let us take a closer look at that description. In 1826, Westfeld published a few pages [West1826] describing the device shown in fig. 11. It looks quite similar to a pair of compasses, but instead of a spike there is a "mileage"-counter.

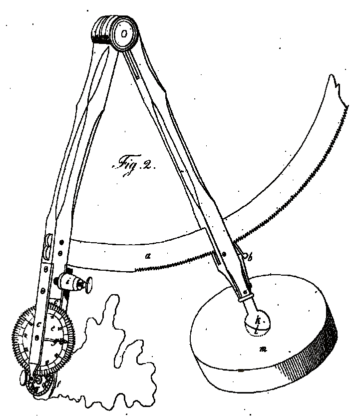


Fig. 11 - Westfeld's Ringmesser [West1826]

The counter measures the arc length tangential direction and not in the radius direction. That means that it measures the arc length of a section of a ring.

Westfeld did not use small ring elements $\Delta r \; r \; \Delta\varphi$, but the formula:

$$A = \sum_i r \varphi_i \Delta r$$

where $r \; \varphi_\text{i}$ is the arc length of the ring sections. Westfeld did not even use infinitesimally thin rings according to the formula:

$$A = \int r \; \varphi \; \mathrm{d}r$$

So Westfeld's Ringmesser is only capable of obtainig approximate values for the area. Evidently, Gierer used a completely different approach from Westfeld's for measuring the area a region. Instead of re-inventing the Ringmesser, Gierer invented an entirely new device with a much more complicate mechanism.

Should we blame Gierer for misunderstanding what Bauernfeind wrote about Westfeld? Definitely not. Other readers also misunderstood the available sparse information but did not come up with an original solution:

Baxandall for example, for the catalogue of the collections in the London science museum, writes: "About 1856 a planimeter of the polar type, in which the recording wheel, kept in the required position by means of a guiding curve, rolled on the paper, was designed by Gierer of Fürth."[Baxa1975] "About 1856" probably refers to an article which was published by Jacob Amsler in that year [Amsl1856A].

Amsler decribes the Gierer planimeter correctly, but together with two other devices from Decher and Bouniakovsky. So the recording wheel rolling on the paper mentioned by Baxandall is from Decher or Bouniakovsky but not a feature of Gierers planimeter.

In 1911, Willers describes Westfeld's Ringmesser correctly, but claims that Gierer's planimeter uses the same principle [Will1911], which is wrong. However it is exactly what Gierer claimed to have done.

To complicate things further: Gierer's "polar coordinate planimeter" is not to be confused with the "polar planimeter" that came into use a few years later.

References

[Amsl1856A] AMSLER, Jakob: *Über die mechanische Bestimmung des Flächeninhaltes, der statischen Momente und der Trägheitsmomente ebener Figuren insbesondere über einen neuen Planimeter*. Beck, 1856

[Amsl1856B] AMSLER, Jakob: Über das Polar-Planimeter. In: *Dinglers Polytechnisches Journal*, 1856

[Baue1853] BAUERNFEIND, Carl M.: Die Planimeter von Ernst, Wetli und Hansen, welche den Flächeninhalt ebener Figuren durch das Umfahren des Umfangs angeben. In: Polytechnischer Verein für das Königreich Bayern, Ed.: *Kunst- und Gewerbeblatt*. München, 1853

[Baxa1975] BAXANDALL, D and PUGH, Jane*: Calculating machines and instruments*. Science Museum, 1975

[Fisc2002] FISCHER, Joachim: Instrumente zur Mechanischen Integration II. In: B. Weiss and A. Schürmann, Eds.*: Chemie - Kultur – Geschichte*. Berlin, 2002

[Fron1887] FRONMÜLLER, Georg Tobias Christoph: *Chronik der Stadt Fürth*. 2. Auflage, Fürth, 1887

[Gier1854] GIERER, Johann, *Entwurf eines Planimeters, mit welchem man den Quadratinhalt ebener Figuren nach Kreisausschnittelementen oder auch nach Ringelementen ausmessen kann.* Jul. Volkhard, Fürth, 1854

[Hunä1864] HUNÄUS, Georg Christian: *Die geometrischen Instrumente der Gesammten praktischen Geometrie, deren Theorie, Beschreibung und Gebrauch.* Hanover, Carl Rümpler, 1864

[Schw1968] SCHWAMMBERGER, Adolf: *Fürth von A bis Z. Ein Geschichtslexikon.* Stadtverwaltung Fürth, 1968

[Vett1856] VETTER, Eduard: *Statistisches Hand- und Addreßbuch von Mittelfranken im Königreich Bayern.* Brügelsche Offizin, Ansbach, 1856

[Vett1864] VETTER, Eduard: *Statistisches Hand- und Addreßbuch von Mittelfranken im Königreich Bayern.* Brügelsche Offizin, Ansbach, 1856

[West1826] WESTFELD, O.: *Der Ringmesser, ein Instrument, um Flächen zu berechnen.* Göttingen, 1826

[Will1911] WILLERS, Theodor: *Zur Geschichte der geographischen Flächenmessung seit der Einführung des Planimeters* - Inauguraldissertation zur Erlangung der Doktorwürde der Hohen Philosophischen Fakultät der Georgs-Augustus-Universität zu Göttingen. Gotha : Justus Perthes, 1911